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SELF-DEVELOPED FORMULAS TO YIELD PRIMITIVE PYTHAGOREAN TRIPLES

More than 2000 years ago Pythagoras proved that the area of the square of the hypotenuse on a right triangle equals the sum of the area of the squares of the legs. This relationship is written in this manner:

$$a^2 + b^2 = c^2 \qquad \text{Equation (1)}$$

where a, b, and c are the three sides which form a right triangle.

The purpose of this paper is to show how to develop and prove a set of formulas giving all sets of integral numbers which: (1) can be the three sides of a right triangle, and (2) are irreducible (have no factor in common). These sets of numbers are primitive Pythagorean triples. A set of rules is also developed which shows whether or not three given integers are a primitive Pythagorean triple. When a,b, and c are integers they may be called a Pythagorean triple, and if these integers are irreducible they are called primitive. A Pythagorean triple is 9,12,15, while a primitive Pythagorean triple is 3, 4, 5.

In order to develop and prove the set of formulas, a study has been made as follows:

A. <u>Classified Groups of Triples</u>

Use a table of squares and good estimating to find and classify several triples. The triples fall into either one of two classes where the difference (the x-value) between the hypotenuse integer c, and the generally larger leg a is either Class A (an odd number squared, i.e. 1, 9, 25, ...) or Class B (one half an even number squared, i.e. 2, 8, 18, ...) These triples are classified in Table 1 according to their x-value. "N_1" represents an x-value for both Classes A and B and "N_2" represents an entire triple within a given x-value.

TABLE 1

N_1 ↓		N_2 →	1	2	3	4	5
1	x=1	c=	5	13	25	41	61
		a=	4	12	24	40	60
		b=	3	5	7	9	11
2	x=9	c=	17	29	45	65	89
		a=	8	20	36	56	80
		b=	15	21	27	33	39
3	x=25	c=	37	53	73	97	125
		a=	12	28	48	72	100
		b=	35	45	55	65	75
1	x=2	c=	5	17	37	65	101
		a=	3	15	35	63	99
		b=	4	8	12	16	20
2	x=8	c=	13	29	53	85	125
		a=	5	21	45	77	117
		b=	12	20	28	36	44
3	x=18	c=	25	45	73	109	153
		a=	7	27	55	91	137
		b=	24	36	48	60	72

B. "X-value" Proof

By definition $x = c - a$, thus $a = c - x$, and by substituting in the Pythagorean Theorem, $a^2 + b^2 = c^2$ and solving for b,

$$(c - x)^2 + b^2 = c^2$$

$$c^2 - (c^2 - 2cx + x^2) = b^2$$

$$2cx - x^2 = b^2$$

$$x(2c - x) = b^2 \qquad \text{Equation (2)}$$

$$\sqrt{x(2c - x)} = b \qquad \text{Equation (2a)}$$

Now in order for "b" to be an integer, Equation 2a shows that $x(2c - x)$ must be a perfect square, and if $x(2c - x)$ is a perfect square it must consist of one or more factors, each squared. For example, if $b = 36$:

$$36 = 6^2 = (2\cdot3)^2 = 2^2\cdot3^2,$$

where 2 and 3 are both squared factors. Other examples can be similiarly proved. From Equation 2 above, x is a factor of b^2, but x^2 must also be a valid factor. Thus the other part of x^2 must be contained within $(2c - x)$. Divide x out and the result is: $x^2\frac{(2c - x)}{x} = b^2$, Equation (3)

where x^2 and $\frac{(2c - x)}{x}$ are each squared factors of b^2. But if you solve $x^2 \frac{(2c - x)}{x}$ for b, you get: $\sqrt{x^2}\sqrt{\frac{2c - x}{x}} = b$. Equation (3a)

The difference x may always be made unity by division by itself; and this division also reduces b.. If b is reducible, c and a are also reducible. An example will show this reduciblity clearly: First , let $x = 3$ and substitute into Equation 3a : $\sqrt{3^2}\sqrt{\frac{2c - 3}{3}} = b$. Second, find a value for c making b an integer: when $c = 15$, $b = 9$. Third, find a : $a = c - x = 15 - 3 = 12$. 15, 12, and 9 may be divided by 3, the x-value, yielding 5, 4, 3. The difference $x = c - a = 5 - 4 =$ unity.

Two groups have x-values for which the difference can never be reduced to unity. The first group consists of x_o --an odd number squared. In these cases x_o already equals a squared factor of b, and therefore $(2c-x)$ is also a squared factor. Substitute directly into Equation 2a and the result is: $\sqrt{x_o} \cdot \sqrt{2c - x_o} = b$, Equation (4)

where both $\sqrt{x_o}$ and $\sqrt{2c - x}$ are integers. Equation 4 cannot be reduced like Equation 3a because $\sqrt{x_o}/x_o$ is never an integer (except unity, which is an odd-number squared.

The second group consists of x_e, - one half an even number squared. First, alter Equation 2 by multiplying and dividing by 2;

substitute in x_e and the result is: $2x_e(c - x_e/2) = b^2$. This makes $2x_e$ a squared factor of b, and thus $(c - x_e/2)$ is also a squared factor. Solving for b, the result is $\sqrt{2x_e} \cdot \sqrt{c - x_e/2} = b$ Equation (5)

where both $\sqrt{2x_e}$ and $\sqrt{c - x_e/2}$ are integers. Therefore, Equation 5, like Equation 4, is not reducible because $\sqrt{2x_e}/x_e$ is never an integer (except unity).

Use deductive reasoning to prove the hypotenuse of each triple is and odd integer if the triple is primitive.

1. If x is even, then b is even, - Equation 2.
2. If c is even and x is even, then a is even, $(a = c - x)$.
3. If c is even and x is even then both a and b are also even.
4. If all the sides are even then they are all reducible by at least 2, always making the hypotenuse odd.
5. If x is odd, then c must be odd, in order to produce integral values for b, - Equation 2a.

C. Relationships Within Triples

Use TABLE I to find definite relationships within the triples, which may be represented by a formula. These formulas may then be considered representative of all triples since, by the "x-value" proof, TABLE I contains the beginning of all types of triples. Use experimentation and arithmetic progression to develop the relationship formulas, and prove them by mathematical induction. N_1 and N_2 are any two positive integers. These relationships are contained in TABLE II.

TABLE II

X_1

$n_2 \rightarrow$	all n_2	1	2	3	1	2	3
$n_1 \downarrow$	x	$(b-x)$	$(b-x)$	$(b-x)$	$(c-b)$	$(c-b)$	$(c-b)$
1	1	2	4	6	2	8	18
2	9	6	12	18	2	8	18
3	25	10	20	30	2	8	18
	$x = 1 + 4n_1(n_1 - 1)$	$b - x = 2n_2(2n_1 - 1)$			$c - b = 2n_2^2$		

Y_1

$n_2 \rightarrow$	all n_2	1	2	3	1	2	3
$n_1 \downarrow$	x	$(b-x)$	$(b-x)$	$(b-x)$	$(c-b)$	$(c-b)$	$(c-b)$
1	2	2	6	10	1	9	25
2	8	4	12	20	1	9	25
3	18	6	18	30	1	9	25
	$x = 2n_2^2$	$b - x = 2n_1(2n_2 - 1)$			$c - b = 1 + 4n_2(n_2 - 1)$		

D. Final Formula Development

Combine certain relationship formulas for each side of the triangle to obtain a set of formulas for both Class A and Class B which contain N_1 and N_2.

X

$$c = (x) + (b-x) + (c-b) = [1+4n_1(n_1-1)] + [2n_2(2n_1-1)] + [2n_2^2]$$

$$\underline{c = 1 + 2n_2(n_2-1) + 4n_1(n_1+n_2-1)}$$

$$a = c - x = (b-x) + (c-b) = [2n_2(2n_1-1)] + [2n_2^2]$$

$$\underline{a = 2n_2(2n_1+n_2-1)}$$

$$b = (x) + (b-x) = [1+4n_1(n_1-1)] + [2n_2(2n_1-1)]$$

$$\underline{b = 1 - 2n_2 + 4n_1(n_1+n_2-1)}$$

Y

$$c = (x) + (b-x) + (c-b) = [2n_1^2] + [2n_1(2n_2-1)] + [1+4n_2(n_2-1)]$$

$$\underline{c = 1 + 2n_1(n_1-1) + 4n_2(n_1+n_2-1)}$$

$$a = c - x = (b-x) + (c-b) = [2n_1(2n_2-1)] + [1+4n_2(n_2-1)]$$

$$\underline{a = 1 - 2n_1 + 4n_2(n_1+n_2-1)}$$

$$b = (x) + (b-x) = [2n_1^2] + [2n_1(2n_2-1)]$$

$$\underline{b = 2n_1(n_1+2n_2-1)}$$

Compare the two sets of final formulas X and Y, by substituting N_2 for N_1 and N_1 for N_2 into formulas X. The results are equal to formulas Y. This can be done because both N_1 and N_2, by definition, can be any natural number, and their only purose was to help organize the way in which triples were to be written. Both sets of formulas yield the same solutions, and one of the sets may be disregarded--disregard group X. This equality of groups may also be noted in Table 1 by comparing one row of group X with its corresponding column of group Y.

By using algebra it is possible to expand and simplify each formula of group Y.

$$
\begin{aligned}
c &= 1 + 2n_1(n_1-1) + 4n_2(n_1+n_2-1) \\
&= 1 + 2n_1^2 - 2n_1 + 4n_1n_2 + 4n_2^2 - 4n_2 \\
&= [(n_1^2 - 2n_1 + 1) + (4n_1n_2 - 4n_2) + 4n_2^2] + n^2 \\
&= [(n_1-1)^2 + 4n_2(n_1-1) + (2n_2)^2] + n_1^2 \\
&= [(n_1-1)^2 + (2n_2)]^2 + n_1^2 = \underline{(2n_2+n_1-1)^2 + n_1^2}
\end{aligned}
$$

$$
\begin{aligned}
a &= 1 - 2n_1 + 4n_2(n_1+n_2-1) \\
&= 1 - 2n_1 + 4n_2n_1 + 4n_2^2 - 4n_2 \\
&= [(n_1^2 - 2n_1 + 1) + (4n_2n_1 - 4n_2) + 4n^2] - n_1^2 \\
&= [(n_1-1)^2 + 4n_2(n_1-1) + (2n_2)^2] - n_1^2 \\
&= [(n_1-1) + (2n_2)]^2 - n_1^2 = \underline{(2n_2+n_1-1)^2 - n_1^2}
\end{aligned}
$$

$$
b = \underline{2n_1(2n_2+n_1-1)}
$$

$$
\boxed{\begin{aligned}
c &= (2n_2+n_1-1)^2 + n_1^2 \\
a &= (2n_2+n_1-1)^2 - n_1^2 \\
b &= 2n_1(2n_2+n_1-1)
\end{aligned}}
$$

E. Table of Triples

Use the developed formulas ~~txx~~ above to make a table which contains all primitive triples with a hypotenuse integer of less than 1000. Table III contains these triples, and it may be noted that the even-numbered b is always divisible by 4 and that the hypotenuse c minus one is also divisible by 4.

Table 4

~~E.~~ Solutions

	5	13	17	25	29	37	41	53	65	65	73	85	85	89	97	101	109	113	117
a	3	5	15	7	21	35	9	45	63	33	55	77	13	39	65	99	91	15	45
b	4	12	8	24	20	12	40	28	16	56	48	36	84	80	72	20	60	112	108

c =	125	137	145	145	149	153	157	169	173	181	185	185	193	197	205	221
a =	117	105	17	143	51	137	85	119	165	19	153	57	95	195	187	21
b =	44	88	144	24	140	72	132	120	52	180	104	176	168	28	84	220
c =	221	225	229	233	241	245	257	261	265	265	269	277	281	289	293	305
a =	171	63	221	105	309	147	255	189	247	23	69	115	231	161	285	207
b =	140	216	60	208	120	196	32	180	96	264	260	252	160	240	68	224
	305	313	317	325	325	333	337	349	353	365	365	369	373	377	377	377
	273	25	75	253	323	315	175	299	225	27	357	81	275	345	175	135
	136	312	308	204	36	108	288	180	272	364	76	360	252	152	288	352
	389	397	401	405	409	421	425	425	433	445	445	449	457	461	477	481
	189	325	399	243	391	29	87	297	145	203	437	351	425	261	405	21
	325	399	40	324	120	420	416	304	408	396	84	280	168	380	252	480
	481	485	485	493	493	505	505	509	521	533	533	541	545	545	549	557
	319	483	93	155	475	217	377	459	279	435	525	341	513	33	99	165
	360	44	476	468	132	456	336	220	440	308	92	420	184	544	540	532
	565	565	569	577	585	585	593	601	605	613	617	625	629	629	637	641
	396	276	520	48	504	144	368	240	484	612	608	336	100	460	588	200
	403	493	231	575	297	567	465	551	363	35	105	527	621	429	245	609
	653	657	661	673	677	685	685	689	689	697	697	701	709	725	725	745
	315	495	589	385	675	37	667	561	111	185	455	651	259	333	627	497
	572	432	300	552	52	684	156	400	680	672	528	260	660	644	364	629
	745	757	761	765	765	769	773	773	785	785	793	793	797	801	809	821
	713	595	39	117	693	481	725	195	273	783	665	775	555	351	759	429
	216	468	760	756	324	600	108	748	736	56	432	168	572	720	280	700
	829	833	841	845	845	845	853	857	865	865	873	877	881	901	901	905
	629	735	41	597	837	123	205	825	287	703	585	805	369	451	899	663
	540	392	840	676	116	836	828	232	816	504	648	348	800	780	60	616
	909	925	925	929	937	941	949	949	953	965	965	977	981	985	985	997
	891	43	533	129	215	741	301	851	705	957	387	945	819	697	473	925
	180	924	756	920	912	580	900	420	728	124	884	248	540	696	864	372

F. Rules For Triples

The Pythagorean Theorem can be used to determine if any three given integers are a triple. For example 543, 458 and 311 is a triple if $543^2 = 458^2 + 311^2$. This method is laborious and several simple rules can instead be used to tell if the three numbers are a triple.

THE THREE NUMBERS ARE NOT A TRIPLE IF:

Rule 1. By observation, the numbers will obviously not satisfy the equation.

EXAMPLE: 1689, 54, and 36

Rule 2. Only one number is odd or all three numbers are odd.

EXAMPLE: 84, 64, 55 or 85, 63, 55

PROOF: $\text{Odd}^2 \stackrel{?}{=} \text{even}^2 + \text{even}^2$ (and) $\text{Odd}^2 \stackrel{?}{=} \text{odd}^2 + \text{odd}^2$

$\text{Odd} \stackrel{?}{=} \text{even} + \text{even}$ $\qquad$ $\text{odd} \stackrel{?}{=} \text{odd} + \text{odd}$

$\text{Odd} \neq \text{even}$ $\qquad$ $\text{odd} \neq \text{even}$

Rule 3. The largest number is even and both of the smaller numbers are odd.

EXAMPLE: 96, 71, 37

PROOF: see page 4 at the end of Section B.

Rule 4. When three even numbers are reduced b y multiples of two down to one or more odd numbers, either Rule 2 or Rule 3 is true.

EXAMPLE: 154, 118, 98 reduces to 77, 59, 49; this violates Rule 2.

Rule 5. The last digit of the sum of the squares of the last digit of each of the two smallest numbers does not equal the last digit of the square of the last digit of the largest number.

EXAMPLE: $5897\underline{6}$, $4532\underline{8}$, $3114\underline{9}$; $8^2 + 9^2 \stackrel{?}{=} 6^2$, $6\underline{4} + 8\underline{1} \stackrel{?}{=} 3\underline{6}$;

$4 + 1 \stackrel{?}{=} 6$, $5 \neq 6$

PROOF: If the last digit of each of the smaller numbers squared does not equal the last digit of the largest number squared, then there is no valid solution, no matter what the other digits are.

Rule 6. When reducdd to two odds an d an even, the difference between the largest number and the even number is not a perfect square.

EXAMPLE: 631, 484, 397; $631 - 484 = 147 \neq$ perfect square

PROOF: Section B, page 3

Rule 7. When reduced to two odds and an even, one half the difference between the two odd numbers is not a perfect square.

EXAMPLE: 631, 484, 395; $\frac{631 - 395}{2} = 118 \neq$ perfect square

PROOF: Section B, pages 3 and 4

Rule 8. When the difference between the two odd numbers and the even number ~~is~~ are reduced to lowest possible terms, the square of the reduced difference plus the square of the reduced even number does not equal the largest number. NOTE: Make use of Rule 5 whenever helpful.

EXAMPLE: 631, 484, 395; $\frac{631 - 395}{484} = \frac{236}{484} = \frac{59}{121}$, $59^2 + 121^2 \overset{?}{=} 631$,

but by Rule 5, $59^2 + 121^2 \neq 631$

PROOF: Using the final formulas and letting $v = 2n_2 + n_1 - 1$ and $w = n_1$, the result is $\frac{c - a}{b} = \frac{(v^2 + w^2) - (v^2 - w^2)}{2wv} = \frac{2w^2}{2wv} = \frac{w}{v}$, $v^2 + w^2 = c$

G. Conclusion

Several triples were found by trial and error. This involved testing groups of three integers in the ~~formulas~~ formula, $c^2 = a^2 + b^2$ until a sufficient number of triples was found.

The triples were proved to be restricted to two classes, where the difference between the two largest integers is (A) an odd square or (B) one-half an even square.

Common relationships between consecutive triples were found and served as a guide for predicting other triples.

These various relationships were combined and this gave a complete formula for each side of the triangle.

These formulas were simplified by algebra and are in terms of two unknown positive integers. A set of rules was then added to determine if three given numbers are a triple or not.

~~Nat.~~ [redacted] A.

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$$a = \frac{b^2 - x^2}{2x}$$